

Homothetic Vectors of Bianchi Type I Spacetimes in Lyra Geometry and General Relativity

Ahmad T Ali^{1,2}, Suhail Khan³ and Azeb Alghanemi¹

1- King Abdulaziz University, Faculty of Science, Department of Mathematics,
PO Box 80203, Jeddah, 21589, Saudi Arabia.

2- Mathematics Department, Faculty of Science, Al-Azhar University,
Nasr City, 11884, Cairo, Egypt.

3- Department of Mathematics, University of Peshawar, Peshawar,
Khyber Pakhtoonkhwa, Pakistan.
E-mail: suhail_74pk@yahoo.com

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Abstract

In this paper Bianchi type I spacetimes are completely classified by their homothetic vectors in the context of Lyra geometry. The non-linear coupled Lyra homothetic equations are obtained and solved completely for different cases. In some cases, Bianchi type I spacetimes admit proper Lyra homothetic vectors (LHVs) for special choices of the metric functions, while there exist other cases where the spacetime under consideration admits only Lyra Killing vectors (LKVs). In all the possible cases where Bianchi type I spacetimes admit proper LHVs or LKVs, we obtained homothetic and Killing vectors for Bianchi type I spacetimes in general relativity by taking the displacement vector of Lyra geometry as zero. Matter collineation symmetry is explained by taking the matter field as a perfect fluid. The obtained proper LHVs and LKVs are used in matter collineation equations and a barotropic equation of state having $\rho(t) = \gamma p(t)$, $0 \leq \gamma \leq 1$ form is always obtained when the displacement vector is considered as a function of t or treated as a constant.

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1 Introduction

Solutions of Einstein field equations (EFE's) are obtained through symmetries in general relativity [1, 2, 3, 17, 18, 36]. Symmetries of a spacetime play a central role to classify these

invariant solutions [14]. Lie symmetries of various geometrical and physical quantities in general relativity have been studied by many authors, e.g. see [8, 32]. Different kinds of symmetries have been extensively studied and various aspects of physical and geometrical interests have been discussed (see for example [5, 8]). General relativity is based upon Riemannian geometry while another equivalent description of gravity known as teleparallel theory is based upon Weitzenbock geometry [35] and symmetries of various kinds are also explored there [12, 13, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. Isometries (also called Killing vectors) of a spacetimes are those vectors which carries the metric tensor invariantly when dragged through Lie transport. Different Lie symmetries have been used to construct new solutions of EFE's [11]. Conformal Killing vectors always satisfy equation of the form

$$\mathcal{L}_X g_{ij} = 2\alpha g_{ij}, \quad (1)$$

where g_{ij} denotes the metric tensor components, X represents Killing vector field, \mathcal{L}_X denotes Lie derivative along X and α is conformal factor depending upon spacetime coordinates. For a constant α , X becomes a homothetic vector (HV). If this constant is **non-zero**, X is called *proper HV* and for $\alpha = 0$, X becomes a *Killing vector field*. It is to remind that spacetime curvature has a great influence over CKVF's. When a spacetime remains non-conformally flat, the maximum dimension of the Lie algebra of CKVF's is seven [9, 10], while for conformally flat spacetimes it is fifteen. Symmetries can be applied to explore the kinematical, dynamical or geometrical properties of a spacetime manifold [33]. Conformal symmetry can be applied to different spacetime manifolds to obtain information about rotation, shear or expansion [16]. Symmetries in general and conformal symmetry in particular have shown an important role in obtaining new solutions of EFE's [4]. At the geometric level symmetries make possible coordinate choice to simplify the metric [34]. Different modifications of Riemannian geometry are proposed to tackle the problems such as unification of the laws of electromagnetism and gravitation, coupling of matter fields with gravitational field and singularities of standard cosmology. One such modification is given by Lyra [15] in 1951 by introducing a gauge function. Riemannian geometry is also modified by Sen [19, 20] and Dunn [21]. Their proposed scalar tensor theory resembles to EFE's. Recently, Lie derivative for tensors along vector fields on Lyra manifold is introduced in [6, 7] as

$$\mathcal{L}_X^L g_{\mu\nu} = \frac{1}{x^o} (g_{\mu\rho} X_{,\nu}^\rho + g_{\rho\nu} X_{,\mu}^\rho) + (g_{\mu\rho} \Gamma_{\lambda\nu}^\rho + g_{\rho\nu} \Gamma_{\mu\lambda}^\rho) X^\lambda, \quad (2)$$

where \mathcal{L}_X^L denotes the Lyra Lie derivative of the Lyra geometry, x^o is a gauge function of the spacetime coordinates and $\Gamma_{\mu\nu}^\rho$ is Lyra affine connection given by:

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{x^o} \{\lambda_{\mu\nu}\} + \frac{1}{2} (\delta_\mu^\lambda \phi_\nu + \delta_\nu^\lambda \phi_\mu - g_{\mu\nu} \phi^\lambda), \quad (3)$$

where $\delta_\mu^\lambda = \text{diag}(1, 1, 1, 1)$ is the Kronicker delta, $\{\lambda_{\mu\nu}\}$ is the Riemannian connection and $\phi_\mu = (\theta(t), 0, 0, 0)$ is displacement vector with a displacement function $\theta(t)$.

Homothetic symmetries in Lyra geometry have been discussed for only few spacetimes by Gad and Alofi [6] and Gad [7]. They obtained only special solutions of their homothetic equations. Keeping in mind the wide range applications of Lie symmetries at kinematics, dynamics and geometric levels in general theory of relativity, our aim is to completely classify Bianchi type I spacetimes by its LHVs. Also, proper LHVs and LKVs will be used into matter collineations equations to obtain a barotropic equation of state of the form $\rho(t) = \gamma p(t)$, $0 \leq \gamma \leq 1$ for the perfect fluid matter field.

It is important to highlight that this paper supersedes [38] by improving some results and by introducing a new section of Matter Collineation. Some notations are also changed in this new version. The remainder part of this paper is organized as follows: In section 2, non-linear coupled Lyra homothetic equations are introduced. Those equations are integrated directly to obtain a general solution. During the process some integrability conditions are also obtained, which are also solved. In section 3, matter collineations equations are obtained and solved to get a barotropic equation of state. In the last section a conclusion to the paper is given.

2 Lyra homothetic equations and their solutions

We consider Bianchi type I spacetimes in the convention coordinates $(x^0 = t, x^1 = x, x^2 = y, x^3 = z)$, in the form

$$ds^2 = -dt^2 + U^2(t) dx^2 + W^2(t) (dy^2 + dz^2), \quad (4)$$

where U and W are no where zero functions of t only. If we choose the normal gauge $x^o = 1$, the non-zero Lyra connections for spacetime (4) can be computed through Eq.(3) as follows

$$\begin{cases} \Gamma_{00}^0 = \frac{\theta(t)}{2}, & \Gamma_{01}^1 = \frac{U'(t)}{U(t)} + \frac{\theta(t)}{2}, & \Gamma_{02}^2 = \Gamma_{03}^3 = \frac{W'(t)}{W(t)} + \frac{\theta(t)}{2}, \\ \Gamma_{11}^0 = U(t)U'(t) + \frac{U^2(t)\theta(t)}{2}, & \Gamma_{22}^0 = \Gamma_{33}^0 = W(t)W'(t) + \frac{W^2(t)\theta(t)}{2}, \end{cases} \quad (5)$$

where prime denotes the derivatives with respect to t . Using the metric (4) and Lyra connections (5) in the Lyra homothetic equation (1) along with Lyra Lie derivative (2) yields the system of ten partial differential equations in four unknowns X^0, X^1, X^2 and X^3 as the following:

$$2X_{,t}^0 + \theta(t)X^0 = 2\alpha, \quad (6)$$

$$X_{,x}^0 = U^2 X_{,t}^1, \quad (7)$$

$$X_{,y}^0 = W^2 X_{,t}^2, \quad (8)$$

$$X_{,z}^0 = W^2 X_{,t}^3, \quad (9)$$

$$2U(t) X_{,x}^1 + [\theta(t)U(t) + 2U'(t)] X^0 = 2\alpha U(t), \quad (10)$$

$$X_{,y}^1 U^2 + W^2 X_{,x}^2 = 0, \quad (11)$$

$$X_{,z}^1 U^2 + W^2 X_{,x}^3 = 0, \quad (12)$$

$$2W(t) X_{,y}^2 + [\theta(t)W(t) + 2W'(t)] X^0 = 2\alpha W(t), \quad (13)$$

$$X_{,z}^2 + X_{,y}^3 = 0, \quad (14)$$

$$2W(t) X_{,z}^3 + [\theta(t)W(t) + 2W'(t)] X^0 = 2\alpha W(t), \quad (15)$$

We will integrate some suitable equations from the above equations to get vector field components X^0 , X^1 , X^2 and X^3 . The whole process is explained as follows: Differentiating Eq. (8) with respect to z , Eq. (9) with respect to y and Eq. (14) with respect to t to obtain a relation

$$X_{,yz}^0 = X_{,tz}^2 = X_{,ty}^3 = 0. \quad (16)$$

Similarly, differentiating Eqs. (11), (12) and (14), with respect to z , y , x , respectively and simplifying, we get

$$X_{,yz}^1 = X_{,xz}^2 = X_{,xy}^3 = 0. \quad (17)$$

By applying similar techniques between Eqs. (8), (9), (11), (12), (13), (14) and (15), one gets

$$X_{,yy}^i - X_{,zz}^i = X_{,yy}^j + X_{,zz}^j = 0, \quad i = 0, 1, \quad j = 2, 3. \quad (18)$$

Integrating Eqs. (16) and (17), substituting the results in (18), integrating again (18) and solving with the help of Eqs. (8), (9), (11), (12) and (14), we get

$$\begin{cases} X^0 = F^0 + W^2(t) \left[y F^2 + z F^3 + (y^2 + z^2) F^4 \right]_{,t}, \\ X^1 = \alpha x + F^1 - \frac{W^2(t)}{U^2(t)} \left[y F^2 + z F^3 + (y^2 + z^2) F^4 \right]_{,x}, \\ X^2 = F^2 + (\alpha + 2F^4) y + d_1 z + d_2 (y^2 - z^2) + 2d_3 y z, \\ X^3 = F^3 - d_1 y + (\alpha + 2F^4) z - d_3 (y^2 - z^2) + 2d_2 y z, \end{cases} \quad (19)$$

$F^i = F^i(t, x)$ are integration functions for all $i = 0, 1, 2, 3, 4$, while d_1 , d_2 and d_3 are constants of integration. Substituting the above vector field components in the system (6)-(15) and making use of *Mathematica Program* we get the following constraints:

$$W(t) [\theta(t)W(t) + 2W'(t)] F_{,t}^i + 4d_i = 0, \quad i = 2, 3, 4, \quad (20)$$

$$[\theta(t)W(t) + 2W'(t)] F^0 + 4W(t) F^4 = 0, \quad (21)$$

$$U(t) [\theta(t)U(t) + 2U'(t)] F_{,t}^i = 2F_{,xx}^i, \quad i = 2, 3, 4, \quad (22)$$

$$[\theta(t)U(t) + 2U'(t)]F^0 + 2U(t)F_{,x}^1 = 0, \quad (23)$$

$$\left(\left[\frac{W(t)}{U(t)} \right] \left[\frac{U(t)}{W(t)} \right]' F^i - F_{,t}^i \right)_{,x} = 0, \quad i = 2, 3, 4, \quad (24)$$

$$F_{,x}^0 = U^2(t)F_{,t}^1, \quad (25)$$

$$[\theta(t)W(t) + 4W'(t)]F_{,t}^i + 2W(t)F_{,tt}^i = 0, \quad i = 2, 3, 4, \quad (26)$$

$$\theta(t)F^0 + 2F_{,t}^0 = 2\alpha, \quad (27)$$

where $d_4 = 0$. We have solved these constrains for different choices of the metric functions and displacement function $\theta(t)$ and the final form of LHV's are obtained. In the following, only final results are listed to avoid lengthy details:

2.1 Proper Lyra Homothetic Vectors For Bianchi Type I Spacetimes

In this section different possibilities for the metric functions and displacement function are explored where Bianchi type I spacetimes admit proper Lyra homothetic vectors. The cases where the spacetimes manifolds admit only Lyra Killing vectors will be discussed in the next sub-section. Details are omitted and the results are written here directly as different possibilities:

Solution (LG1):

$$\left\{ \begin{array}{l} X^0 = \exp \left[- \int \frac{\theta(t)}{2} dt \right] \left(a_1 + a_2 x + a_3 y + a_4 z + \alpha \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right), \\ X^1 = a_6 + \alpha x + a_7 y + a_8 z + a_2 \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt, \\ X^2 = a_9 - a_7 x + \alpha y + a_{10} z + a_3 \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt, \\ X^3 = a_{11} - a_8 x - a_{10} y + \alpha z + a_4 \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt, \end{array} \right. \quad (28)$$

where $U(t) = W(t) = \exp \left[- \int \frac{\theta(t)}{2} dt \right]$ while $\theta(t)$ is an arbitrary function, a_i , $i = 1, 2, \dots, 11$, are arbitrary constants such that $a_5 = \alpha$.

Subtracting Killing vector fields from (28), the proper Lyra homothetic vector is obtained as

$$X = \exp \left[- \int \frac{\theta(t)}{2} dt \right] \left(\int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right) \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}. \quad (29)$$

The generators of Lyra homothetic vector fields (28) can be written as the following:

$$X = \sum_{i=1}^{11} a_i Z_i, \quad (30)$$

where

$$\left\{ \begin{array}{l} Z_1 = \exp \left[- \int \frac{\theta(t)}{2} dt \right] \frac{\partial}{\partial t}, \\ Z_2 = x \exp \left[- \int \frac{\theta(t)}{2} dt \right] \frac{\partial}{\partial t} + \left(\int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right) \frac{\partial}{\partial x}, \\ Z_3 = y \exp \left[- \int \frac{\theta(t)}{2} dt \right] \frac{\partial}{\partial t} + \left(\int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right) \frac{\partial}{\partial y}, \\ Z_4 = z \exp \left[- \int \frac{\theta(t)}{2} dt \right] \frac{\partial}{\partial t} + \left(\int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right) \frac{\partial}{\partial z}, \\ Z_5 = \exp \left[- \int \frac{\theta(t)}{2} dt \right] \left(\int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right) \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}, \\ Z_6 = \frac{\partial}{\partial x}, \quad Z_7 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad Z_8 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \\ Z_9 = \frac{\partial}{\partial y}, \quad Z_{10} = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \quad z_{11} = \frac{\partial}{\partial z}. \end{array} \right. \quad (31)$$

The generators above form close Lie algebra structure with non-zero Lie brackets, given by:

$$\left\{ \begin{array}{l} [Z_1, Z_2] = Z_6, \quad [Z_1, Z_3] = Z_9, \quad [Z_1, Z_4] = Z_{11}, \quad [Z_1, Z_5] = Z_1, \\ [Z_2, Z_3] = -Z_7, \quad [Z_2, Z_4] = -Z_8, \quad [Z_2, Z_6] = Z_1, \quad [Z_2, Z_7] = -Z_3, \\ [Z_2, Z_8] = -Z_4, \quad [Z_3, Z_4] = -Z_{10}, \quad [Z_3, Z_7] = Z_2, \quad [Z_3, Z_9] = -Z_1, \\ [Z_3, Z_{10}] = -Z_4, \quad [Z_4, Z_8] = Z_2, \quad [Z_4, Z_{10}] = Z_3, \quad [Z_4, Z_{11}] = -Z_1, \\ [Z_5, Z_6] = -Z_6, \quad [Z_5, Z_9] = -Z_9, \quad [Z_5, Z_{11}] = -Z_{11}, \quad [Z_6, Z_7] = -Z_9, \\ [Z_6, Z_8] = -Z_{11}, \quad [Z_7, Z_8] = Z_{10}, \quad [Z_7, Z_9] = -Z_6, \quad [Z_7, Z_{10}] = -Z_8, \\ [Z_8, Z_{10}] = Z_7, \quad [Z_8, Z_{11}] = -Z_6, \quad [Z_9, Z_{10}] = -Z_{11}, \quad [Z_{10}, Z_{11}] = -Z_9 \end{array} \right. \quad (32)$$

Later, in Section 3, the role of the displacement vector θ will be analyzed in determining the Matter Collineation symmetry and in the investigation of barotropic equation of state. In a paper [7], the author claimed that this equation of state never satisfies when θ remains a function of t or a constant. It is claimed that it satisfies only when $\theta = 0$. In order to check whether or not barotropic equation of state satisfies in our case, we need to obtain homothetic vectors when θ is a constant and when $\theta = 0$. These two solutions are obtained in the following:

Solution (GR1): ($\theta = 0$): It is interesting to see that the metric functions are dependent upon the displacement function θ and taking $\theta = 0$, the Bianchi type I spacetime becomes flat and its LHVs reduce to the HVs of general relativity [37], which are given as follows:

$$\left\{ \begin{array}{l} X^0 = a_1 + \alpha t + a_2 x + a_3 y + a_4 z, \\ X^1 = a_6 + a_2 t + \alpha x + a_7 y + a_8 z, \\ X^2 = a_9 + a_3 t - a_7 x + \alpha y + a_{10} z, \\ X^3 = a_{11} + a_4 t - a_8 x - a_{10} y + \alpha z, \end{array} \right. \quad (33)$$

where $U(t) = W(t) = 1$ and $a_i, i = 1, 2, \dots, 11$ are constants such that $a_5 = \alpha$. The proper homothetic vector field is obtained as:

$$X = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}. \quad (34)$$

Solution (LG1): ($\theta = 2\theta_0$): This solution is obtained from the above case (LG1) by taking θ as constant.

$$\left\{ \begin{array}{l} X^0 = (a_1 + a_2 x + a_3 y + a_4 z) e^{-\theta_0 t} + \frac{\alpha}{\theta_0}, \\ X^1 = a_6 + \alpha x + a_7 y + a_8 z + a_2 e^{\theta_0 t}, \\ X^2 = a_9 - a_7 x + \alpha y + a_{10} z + a_3 e^{\theta_0 t}, \\ X^3 = a_{11} - a_8 x - a_{10} y + \alpha z + a_4 e^{\theta_0 t}, \end{array} \right. \quad (35)$$

where $U(t) = W(t) = e^{-\theta_0 t}$, $a_i, i = 1, 2, \dots, 11$, are constants obtained during integration such that $a_5 = \alpha$. In this case the proper Lyra homothetic vector becomes:

$$X = \frac{1}{\theta_0} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}. \quad (36)$$

Solution (LG2):

$$\left\{ \begin{array}{l} X^0 = \exp \left[- \int \frac{\theta(t)}{2} dt \right] \left(a_1 + \alpha \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right), \\ X^1 = a_2 + a_3 x + a_4 y + a_5 z, \\ X^2 = a_6 - a_4 x + a_3 y + a_5 z, \\ X^3 = a_8 - a_5 x - a_7 y + a_3 z, \end{array} \right. \quad (37)$$

where $U(t) = W(t) = \exp \left[- \int \frac{\theta(t)}{2} dt \right] \left(a_1 + \alpha \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right)^{1-\frac{a_3}{\alpha}}$ while $\theta(t)$ is an arbitrary function, α and $a_i, i = 1, 2, \dots, 8$, are arbitrary constants such that $a_3 \neq \alpha$.

Subtracting Killing vector fields from (37), the proper Lyra homothetic vector is obtained as

$$X = \exp \left[- \int \frac{\theta(t)}{2} dt \right] \left(\int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right) \frac{\partial}{\partial t}. \quad (38)$$

Solution (GR2): ($\theta = 0$): It is interesting to see that the metric functions are dependent upon the displacement function θ and taking $\theta = 0$, the LHV's reduce to the HV's of general relativity, as obtained in [37], case (5) and are given as follows:

$$\left\{ \begin{array}{l} X^0 = a_1 + \alpha t, \\ X^1 = a_2 + a_3 x + a_4 y + a_5 z, \\ X^2 = a_6 - a_4 x + a_3 y + a_5 z, \\ X^3 = a_8 - a_5 x - a_7 y + a_3 z, \end{array} \right. \quad (39)$$

where $U(t) = W(t) = (a_1 + \alpha t)^{1-\frac{a_3}{\alpha}}$ and $a_i, i = 1, 2, \dots, 8$ are constants such that $a_3 \neq \alpha$. The proper homothetic vector field is obtained as:

$$X = t \frac{\partial}{\partial t}. \quad (40)$$

Solution (LG2): ($\theta = 2\theta_0$): This solution is obtained from the above case (LG2) by

taking θ as constant.

$$\begin{cases} X^0 = a_1 e^{-\theta_0 t} + \frac{\alpha}{\theta_0}, \\ X^1 = a_2 + a_3 x + a_4 y + a_5 z, \\ X^2 = a_6 - a_4 x + a_3 y + a_5 z, \\ X^3 = a_8 - a_5 x - a_7 y + a_3 z, \end{cases} \quad (41)$$

where $U(t) = W(t) = e^{-\theta_0 t} \left(a_1 + \frac{\alpha}{\theta_0} e^{\theta_0 t} \right)^{1 - \frac{a_3}{\alpha}}$, a_i , $i = 1, 2, \dots, 8$, are constants obtained during integration such that $a_3 \neq \alpha$. In this case the proper Lyra homothetic vector becomes:

$$X = \frac{1}{\theta_0} \frac{\partial}{\partial t}. \quad (42)$$

Solution (LG3):

$$\begin{cases} X^0 = W(t) \left[(a_1 + a_2 y + a_3 z) e^{a_0 x} + (a_4 + a_5 y + a_6 z) e^{-a_0 x} \right. \\ \quad \left. + \alpha \left(\gamma_0 + \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right) \right], \\ X^1 = a_7 - a_0^{-1} \left[(a_1 + a_2 y + a_3 z) e^{a_0 x} - (a_4 + a_5 y + a_6 z) e^{-a_0 x} \right] \\ \quad \times \left(\gamma_0 + \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right)^{-1}, \\ X^2 = a_8 + \alpha y + a_9 z + \left(a_2 e^{a_0 x} + a_5 e^{-a_0 x} \right) \left(\gamma_0 + \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right), \\ X^3 = a_{10} - a_9 y + \alpha z + \left(a_3 e^{a_0 x} + a_6 e^{-a_0 x} \right) \left(\gamma_0 + \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right), \end{cases} \quad (43)$$

where $U(t) = a_0 W(t) \left(\gamma_0 + \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right)$ and $W(t) = \exp \left[- \int \frac{\theta(t)}{2} dt \right]$ while $\theta(t)$ is an arbitrary function and $\alpha, \gamma_0, a_i, i = 0, 1, \dots, 10$ are arbitrary constants such that $a_0 \neq 0$.

Subtracting Killing vector fields from (43), the proper Lyra homothetic vector field is obtained as

$$X = \exp \left[- \int \frac{\theta(t)}{2} dt \right] \left(\gamma_0 + \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right) \frac{\partial}{\partial t} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}. \quad (44)$$

Solution (GR3): ($\theta = 0$): For the above case LG3, the HVFs of general relativity can be obtained by taking $\theta = 0$ as follows:

$$\left\{ \begin{array}{l} X^0 = (a_1 + a_2 y + a_3 z) e^x + (a_4 + a_5 y + a_6 z) e^{-x} + \alpha (\gamma_0 + t), \\ X^1 = a_7 + \left[(a_4 + a_5 y + a_6 z) e^{-x} - (a_1 + a_2 y + a_3 z) e^x \right] (\gamma_0 + t)^{-1}, \\ X^2 = a_8 + \alpha y + a_9 z + \left(a_2 e^x + a_5 e^{-x} \right) (\gamma_0 + t), \\ X^3 = a_{10} - a_9 y + \alpha z + \left(a_3 e^x + a_6 e^{-x} \right) (\gamma_0 + t), \end{array} \right. \quad (45)$$

where $a_0 = 1$, $U(t) = \gamma_0 + t$, $W(t) = 1$ and $\theta(t) = 0$ while $\alpha, \gamma_0, a_i, i = 1, \dots, 10$ are arbitrary constants. To the best of our knowledge, this result is not obtained previously in literature. Subtracting Killing vector fields from (45), the proper homothetic vector of general relativity can be obtained as

$$X = (\gamma_0 + t) \frac{\partial}{\partial t} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}. \quad (46)$$

When the displacement function become constant, that is when $\theta(t)$ is an arbitrary constant equals $2\theta_0$, homothetic vector fields in Lyra geometry corresponding to solution (LG3) takes the form:

Solution (LG3): ($\theta = 2\theta_0$):

$$\left\{ \begin{array}{l} X^0 = e^{-\theta_0 t} \left[(a_1 + a_2 y + a_3 z) e^x + (a_4 + a_5 y + a_6 z) e^{-x} \right] + \alpha (\gamma_0 e^{-\theta_0 t} + \theta_0^{-1}), \\ X^1 = a_7 + \left[(a_4 + a_5 y + a_6 z) e^{-x} - (a_1 + a_2 y + a_3 z) e^x \right] (\gamma_0 + \theta_0^{-1} e^{\theta_0 t})^{-1}, \\ X^2 = a_8 + \alpha y + a_9 z + \left(a_2 e^x + a_5 e^{-x} \right) (\gamma_0 + \theta_0^{-1} e^{\theta_0 t}), \\ X^3 = a_{10} - a_9 y + \alpha z + \left(a_3 e^x + a_6 e^{-x} \right) (\gamma_0 + \theta_0^{-1} e^{\theta_0 t}), \end{array} \right. \quad (47)$$

where $a_0 = 1$, $U(t) = \gamma_0 e^{-\theta_0 t} + \theta_0^{-1}$ and $W(t) = e^{-\theta_0 t}$ while $\alpha, \gamma_0, \theta_0, a_i, i = 1, \dots, 10$ are arbitrary constants.

Subtracting Killing vector fields from (47), the proper Lyra homothetic vector is obtained as

$$X = \left(\gamma_0 e^{-\theta_0 t} + \theta_0^{-1} \right) \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}. \quad (48)$$

Solution (LG4):

$$\begin{cases} X^0 = \left(a_1 + \alpha \int \exp \left[\frac{1}{2} \int \theta(t) dt \right] dt \right) \exp \left[-\frac{1}{2} \int \theta(t) dt \right], \\ X^1 = a_3 + a_2 x, \quad X^2 = a_4 + a_5 y + a_6 z, \quad X^3 = a_7 - a_6 y + a_5 z, \end{cases} \quad (49)$$

where $\alpha, a_i, i = 1, \dots, 7$ are arbitrary constants such that $\alpha \neq 0$ while $\theta(t)$ is an arbitrary function. For different possibilities, the proper Lyra homothetic vectors can be obtained as the following along with the metric functions:

(1): If $a_2 \neq \alpha$ and $a_5 \neq \alpha$, then

$$X = \exp \left[-\frac{1}{2} \int \theta(t) dt \right] \left(\int \exp \left[\frac{1}{2} \int \theta(t) dt \right] dt \right) \frac{\partial}{\partial t}, \quad (50)$$

where

$$U(t) = a_0 \exp \left[-\frac{1}{2} \int \theta(t) dt \right] \left(a_1 + \alpha \int \exp \left[\frac{1}{2} \int \theta(t) dt \right] dt \right)^{1-a_2/\alpha}$$

and

$$W(t) = b_0 \exp \left[-\frac{1}{2} \int \theta(t) dt \right] \left(a_1 + \alpha \int \exp \left[\frac{1}{2} \int \theta(t) dt \right] dt \right)^{1-a_5/\alpha}$$

while a_0 and b_0 are arbitrary constants.

(2): If $a_2 = \alpha$ and $a_5 \neq \alpha$, then

$$X = \exp \left[-\frac{1}{2} \int \theta(t) dt \right] \left(\int \exp \left[\frac{1}{2} \int \theta(t) dt \right] dt \right) \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}. \quad (51)$$

where

$$W(t) = b_0 \exp \left[-\frac{1}{2} \int \theta(t) dt \right] \left(a_1 + \alpha \int \exp \left[\frac{1}{2} \int \theta(t) dt \right] dt \right)^{1-a_5/\alpha}$$

and $U(t) = a_0 \exp \left[-\frac{1}{2} \int \theta(t) dt \right]$ while a_0 and b_0 are arbitrary constants.

(3): If $a_2 \neq \alpha$ and $a_5 = \alpha$, then

$$X = \exp \left[-\frac{1}{2} \int \theta(t) dt \right] \left(\int \exp \left[\frac{1}{2} \int \theta(t) dt \right] dt \right) \frac{\partial}{\partial t} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}. \quad (52)$$

where

$$U(t) = a_0 \exp \left[-\frac{1}{2} \int \theta(t) dt \right] \left(a_1 + \alpha \int \exp \left[\frac{1}{2} \int \theta(t) dt \right] dt \right)^{1-a_2/\alpha}$$

and $W(t) = b_0 \exp \left[-\frac{1}{2} \int \theta(t) dt \right]$ while a_0 and b_0 are arbitrary constants.

(4): If $a_2 = \alpha$ and $a_5 = \alpha$, then

$$X = \exp \left[-\frac{1}{2} \int \theta(t) dt \right] \left(\int \exp \left[\frac{1}{2} \int \theta(t) dt \right] dt \right) \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}. \quad (53)$$

where $U(t) = a_0 \exp \left[-\frac{1}{2} \int \theta(t) dt \right]$ and $W(t) = b_0 \exp \left[-\frac{1}{2} \int \theta(t) dt \right]$ while a_0 and b_0 are arbitrary constants.

Solution (GR4): ($\theta = 0$):

$$\begin{cases} X^0 = a_1 + \alpha t, & X^1 = a_3 + a_2 x, \\ X^2 = a_4 + a_5 y + a_6 z, & X^3 = a_7 - a_6 y + a_5 z, \end{cases} \quad (54)$$

where $U(t) = a_0 (a_1 + \alpha t)^{1-a_2/\alpha}$, $W(t) = b_0 (a_1 + \alpha t)^{1-a_5/\alpha}$ and $\theta(t) = 0$ while $\alpha, a_i, i = 0, 1, \dots, 7$ are arbitrary constants such that $\alpha \neq 0$. After removing a minor typing error in case (4) of [37], one can see that our obtained result is exactly same with their result.

Solution (LG4): ($\theta = 2\theta_0$):

$$\begin{cases} X^0 = a_1 e^{-\theta_0 t} + \alpha \theta_0^{-1}, & X^1 = a_3 + a_2 x, \\ X^2 = a_4 + a_5 y + a_6 z, & X^3 = a_7 - a_6 y + a_5 z, \end{cases} \quad (55)$$

where $U(t) = a_0 e^{-\theta_0 t} (a_1 + \alpha \theta_0^{-1} e^{\theta_0 t})^{1-a_2/\alpha}$ and

$W(t) = b_0 e^{-\theta_0 t} (a_1 + \alpha \theta_0^{-1} e^{\theta_0 t})^{1-a_5/\alpha}$ while a_0 and $b_0, \alpha, a_i, i = 1, \dots, 7$ are arbitrary constants such that $\alpha \neq 0$ while $\theta(t)$ is an arbitrary function.

2.2 Killing Vector Fields

In this sub-section, we are going to list all those cases where the spacetime under consideration does not admit proper LHVs and the HVs are just the KVs in the Lyra

geometry.

Solution (LG5):

$$\left\{ \begin{array}{l} X^0 = \frac{4(a_1 + 2a_2 y + 2a_3 z)}{W_0(t) + 2\theta(t)}, \quad X^1 = a_4, \\ X^2 = a_5 - a_1 y + a_6 z - 2a_3 y z - a_2 \left(y^2 - z^2 - 8 \int \frac{W^{-2}(t)}{W_0(t) + 2\theta(t)} dt \right), \\ X^3 = a_7 - a_6 y - a_1 z - 2a_2 y z + a_3 \left(y^2 - z^2 + 8 \int \frac{W^{-2}(t)}{W_0(t) + 2\theta(t)} dt \right), \end{array} \right. \quad (56)$$

where $U(t) = a_0 \exp \left[-\frac{1}{2} \int \theta(t) dt \right]$, $W(t) = b_0 \exp \left[\frac{1}{4} \int W_0(t) dt \right]$,

$$W_0(t) = \left(4b_1 + \int [\theta^2(t) - 2\theta'(t)] \exp \left[-\frac{1}{2} \int \theta(t) dt \right] dt \right) \exp \left[\frac{1}{2} \int \theta(t) dt \right],$$

while $\theta(t)$ is an arbitrary function, $b_0, b_1, a_i, i = 0, 1, \dots, 7$ are arbitrary constants such that $\alpha = 0, a_0 \neq 0$ and $b_0 \neq 0$. This result shows that for particular metric functions (as given above), Bianchi type I spacetime does not admit proper LHV.

Solution (GR5): ($\theta = 0$):

$$\left\{ \begin{array}{l} X^0 = \frac{a_1 + 2a_2 y + 2a_3 z}{b_1}, \quad X^1 = a_4, \\ X^2 = a_5 - a_1 y + a_6 z - 2a_3 y z - a_2 \left(y^2 - z^2 + \frac{e^{-2b_1 t}}{b_0^2 b_1^2} \right), \\ X^3 = a_7 - a_6 y - a_1 z - 2a_2 y z + a_3 \left(y^2 - z^2 - \frac{e^{-2b_1 t}}{b_0^2 b_1^2} \right), \end{array} \right. \quad (57)$$

where $U(t) = a_0$ and $W(t) = b_0 e^{b_1 t}$ while $\theta(t) = 0$, $b_0, b_1, a_i, i = 0, 1, \dots, 7$ are constants, while $\alpha = 0, a_0 \neq 0$ and $b_0 \neq 0$.

Solution (LG5): ($\theta = 2\theta_0$):

$$\left\{ \begin{array}{l} X^0 = \frac{e^{-\theta_0 t} (a_1 + 2a_2 y + 2a_3 z)}{b_1}, \quad X^1 = a_4, \\ X^2 = a_5 - a_1 y + a_6 z - 2a_3 y z - a_2 \left(y^2 - z^2 + \frac{1}{b_0^2 b_1^2} \exp \left[-\frac{2b_1}{\theta_0} e^{\theta_0 t} \right] \right), \\ X^3 = a_7 - a_6 y - a_1 z - 2a_2 y z + a_3 \left(y^2 - z^2 - \frac{1}{b_0^2 b_1^2} \exp \left[-\frac{2b_1}{\theta_0} e^{\theta_0 t} \right] \right), \end{array} \right. \quad (58)$$

where $U(t) = a_0 e^{-\theta_0 t}$ and $W(t) = b_0 e^{-\theta_0 t} \exp \left[\frac{b_1}{\theta_0} e^{\theta_0 t} \right]$ while $b_0, b_1, a_i, i = 0, 1, \dots, 7$ are constants such that $\alpha = 0, a_0 \neq 0$ and $b_0 \neq 0$.

Solution (LG6):

$$\left\{ \begin{array}{l} X^0 = \frac{8(a_1 + a_2 x + a_3 y + a_4 z)}{W_0(t) + 2\theta(t)}, \\ X^1 = a_5 - 2x(a_1 + a_3 y + a_4 z) + a_6 y + a_7 z \\ \quad - a_2 \left[x^2 - y^2 - z^2 - 8 \int \frac{W^{-2}(t)}{W_0(t) + 2\theta(t)} dt \right], \\ X^2 = a_8 - a_6 x - 2y(a_1 + a_2 x + a_4 z) + a_9 z \\ \quad + a_3 \left[x^2 - y^2 + z^2 + 8 \int \frac{W^{-2}(t)}{W_0(t) + 2\theta(t)} dt \right], \\ X^3 = a_{10} - a_7 x - a_9 y - 2z(a_1 + a_2 x + a_3 y) \\ \quad + a_4 \left[x^2 + y^2 - z^2 + 8 \int \frac{W^{-2}(t)}{W_0(t) + 2\theta(t)} dt \right], \end{array} \right. \quad (59)$$

where $U(t) = W(t) = \exp \left[\frac{1}{4} \int W_0(t) dt \right]$,

$$W_0(t) = \left(4b_1 + \int [\theta^2(t) - 2\theta'(t)] \exp \left[-\frac{1}{2} \int \theta(t) dt \right] dt \right) \exp \left[\frac{1}{2} \int \theta(t) dt \right],$$

while $\theta(t)$ is an arbitrary function, $b_1, a_i, i = 0, 1, \dots, 10$ are arbitrary constants such that $\alpha = 0$.

Solution (GR6): ($\theta = 0$):

$$\left\{ \begin{array}{l} X^0 = \frac{2(a_1 + a_2 x + a_3 y + a_4 z)}{b_1}, \\ X^1 = a_5 - 2x(a_1 + a_3 y + a_4 z) + a_6 y + a_7 z - a_2 \left[x^2 - y^2 - z^2 + \frac{e^{-2b_1 t}}{b_1^2} \right], \\ X^2 = a_8 - a_6 x - 2y(a_1 + a_2 x + a_4 z) + a_9 z + a_3 \left[x^2 - y^2 + z^2 - \frac{e^{-2b_1 t}}{b_1^2} \right], \\ X^3 = a_{10} - a_7 x - a_9 y - 2z(a_1 + a_2 x + a_3 y) + a_4 \left[x^2 + y^2 - z^2 - \frac{e^{-2b_1 t}}{b_1^2} \right], \end{array} \right. \quad (60)$$

where $U(t) = W(t) = e^{b_1 t}$ while $\theta(t) = 0$, $b_1, a_i, i = 0, 1, \dots, 10$ are arbitrary constants such that $\alpha = 0$.

Solution (LG6): ($\theta = 2\theta_0$):

$$\left\{ \begin{array}{l} X^0 = \frac{2e^{-\theta_0 t} (a_1 + a_2 x + a_3 y + a_4 z)}{b_1}, \\ X^1 = a_5 - 2x (a_1 + a_3 y + a_4 z) + a_6 y + a_7 z - a_2 \left(x^2 - y^2 - z^2 + \frac{1}{b_1^2} \exp \left[-\frac{2b_1}{\theta_0} e^{-2b_1 t} \right] \right), \\ X^2 = a_8 - a_6 x - 2y (a_1 + a_2 x + a_4 z) + a_9 z + a_3 \left(x^2 - y^2 + z^2 - \frac{1}{b_1^2} \exp \left[-\frac{2b_1}{\theta_0} e^{-2b_1 t} \right] \right), \\ X^3 = a_{10} - a_7 x - a_9 y - 2z (a_1 + a_2 x + a_3 y) + a_4 \left(x^2 + y^2 - z^2 - \frac{1}{b_1^2} \exp \left[-\frac{2b_1}{\theta_0} e^{-2b_1 t} \right] \right), \end{array} \right. \quad (61)$$

where $U(t) = W(t) = e^{-\theta_0 t} \exp \left[\frac{b_1}{\theta_0} e^{-2b_1 t} \right]$ while $b_1, a_i, i = 0, 1, \dots, 10$ are arbitrary constants such that $\alpha = 0$.

Solution (LG7):

$$\left\{ \begin{array}{l} X^0 = [a_1 f_1(x) + a_2 f_2(x)] \exp \left[-\int \frac{\theta(t)}{2} dt \right], \\ X^1 = a_3 + [a_2 f_1(x) + \gamma_0^{-1} a_1 f_1'(x)] f_3(t), \\ X^2 = a_4 + a_5 z, \quad X^3 = a_6 - a_5 y, \end{array} \right. \quad (62)$$

where $\alpha = 0$, $U(t) = \sqrt{\frac{\gamma_0}{f_3'(t)}} \exp \left[-\int \frac{\theta(t)}{2} dt \right]$, $W(t) = b_0 \exp \left[-\int \frac{\theta(t)}{2} dt \right]$, $\theta(t)$ is an arbitrary function and $b_0, \gamma_0, a_i, i = 1, \dots, 6$ are arbitrary constants such that $\gamma_0 \neq 0$ and $b_0 \neq 0$. The functions $f_1(x)$, $f_2(x)$ and $f_3(t)$ lead to two solutions in the following cases:

Case (1): $f_1(x) = \cos [\gamma_0 x]$, $f_2(x) = \sin [\gamma_0 x]$ and $f_3(t) = \tanh \left[c_0 + \gamma_0 \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right]$, where c_0 is an arbitrary constant.

Case (2): $f_1(x) = \cosh [\gamma_0 x]$, $f_2(x) = \sinh [\gamma_0 x]$ and $f_3(t) = \tan \left[c_0 + \gamma_0 \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right]$, where c_0 is an arbitrary constant.

Solution (GR7): ($\theta = 0$): The HVs of general relativity takes the form (62) with $\theta = 0$ such that: $f_3(t) = \tanh[c_0 + \gamma_0 t]$ in case (1) and $f_3(t) = \tan[c_0 + \gamma_0 t]$ in case (2).

The line elements in two cases become:

$$ds_1^2 = -dt^2 + \cosh^2[c_0 + \gamma_0 t] dx^2 + b_0^2 (dy^2 + dz^2), \quad (63)$$

and

$$ds_2^2 = -dt^2 + \cos^2[c_0 + \gamma_0 t] dx^2 + b_0^2 (dy^2 + dz^2). \quad (64)$$

Also, the non-zero components of the energy momentum tensor $\left(T_{ij} = R_{ij} - \frac{R}{2} g_{ij}\right)$ are $T_{33} = T_{33} = \pm b_0^2 \gamma_0^2$.

Solution (LG7): ($\theta = 2\theta_0$):

$$\begin{cases} X^0 = [a_1 f_1(x) + a_2 f_2(x)] e^{-\theta_0 t}, \\ X^1 = a_3 + [a_2 f_1(x) + \gamma_0^{-1} a_1 f_1'(x)] f_3(t), \\ X^2 = a_4 + a_5 z, \quad X^3 = a_6 - a_5 y, \end{cases} \quad (65)$$

where $W(t) = b_0 e^{-\theta_0 t}$ such that: $f_3(t) = \tanh\left[c_0 + \frac{\gamma_0 e^{\theta_0 t}}{\theta_0}\right]$ and $U(t) = \cosh\left[c_0 + \frac{\gamma_0 e^{\theta_0 t}}{\theta_0}\right]$ in case (1) and $f_3(t) = \tan\left[c_0 + \frac{\gamma_0 e^{\theta_0 t}}{\theta_0}\right]$ and $U(t) = \cos\left[c_0 + \frac{\gamma_0 e^{\theta_0 t}}{\theta_0}\right]$ in case (2).

Solution (LG8):

$$\begin{cases} X^0 = (a_1 - 2b_0 a_2 x) W(t), \\ X^1 = a_3 - b_0 a_1 x + a_2 \left(b_0^2 x^2 + \exp\left[-2b_0 \int \exp\left[\int \frac{\theta(t)}{2} dt\right] dt\right] \right), \\ X^2 = a_4 + a_5 z, \quad X^3 = a_6 - a_5 y, \end{cases} \quad (66)$$

where $\alpha = 0$, $W(t) = \exp\left[-\int \frac{\theta(t)}{2} dt\right]$ and

$U(t) = W(t) \exp\left[b_0 \int e^{\int \frac{\theta(t)}{2} dt} dt\right]$ while $\theta(t)$ is an arbitrary function and $a_i, i = 1, \dots, 6$ are arbitrary constants such that $b_0 \neq 0$.

Solution (GR8): ($\theta = 0$):

$$\begin{cases} X^0 = a_1 - 2b_0 c_0 a_2 x, \\ X^1 = a_3 - b_0 a_1 x + a_2 (b_0^2 x^2 + e^{-2b_0 t}), \\ X^2 = a_4 + a_5 z, & X^3 = a_6 - a_5 y, \end{cases} \quad (67)$$

where $\alpha = 0$, $W(t) = 1$ and $U(t) = e^{b_0 t}$ while a_i , $i = 1, \dots, 6$ are arbitrary constants such that $b_0 \neq 0$.

Solution (LG8): ($\theta = 2\theta_0$):

$$\begin{cases} X^0 = (a_1 - 2b_0 c_0 a_2 x) e^{-\theta_0 t}, \\ X^1 = a_3 - b_0 a_1 x + a_2 \left(b_0^2 x^2 + \exp \left[-\frac{2b_0}{\theta_0} e^{\theta_0 t} \right] \right), \\ X^2 = a_4 + a_5 z, & X^3 = a_6 - a_5 y, \end{cases} \quad (68)$$

where $\alpha = 0$, $W(t) = e^{-\theta_0 t}$ and $U(t) = e^{-\theta_0 t} \exp \left[\frac{b_0}{\theta_0} e^{\theta_0 t} \right]$ while a_i , $i = 1, \dots, 6$ are arbitrary constants such that $b_0 \neq 0$.

3 Matter Collineation

A vector field is said to be a Matter Collineations (MC), if the Lie derivative of energy-momentum tensor vanishes along it, mathematically

$$\mathcal{L}_X T_{ij} = 0. \quad (69)$$

If X is a HV then also $\mathcal{L}_X T_{ij} = 0$. Thus every HV is a MC also but converse does not hold in general. The authors of the paper [7], claimed that in Lyra geometry, barotropic equation of state

$$\rho(t) = \gamma p(t), \quad (70)$$

is never satisfied when θ is taken as function of t or a constant. They claimed that Eq. of the form (70) satisfies only when $\theta = 0$. In order to check whether or not Eq. (70) satisfies in our cases, we shall try to establish a relation between the density and pressure.

For this purpose we take the matter field for spacetime under consideration as a perfect fluid, that is, taking energy-momentum tensor of the form

$$T_{ij} = (\rho + p) u_i u_j - p g_{ij}, \quad (71)$$

where, for the spacetime (4), the four-velocity vector is taken as $u^i = (1, 0, 0, 0)$, $u^i u_i = -1$. The non-zero components of the energy-momentum tensor (71) is given by;

$$T_{00} = \rho(t), \quad T_{11} = U^2(t) p(t), \quad T_{22} = T_{33} = W^2(t) p(t). \quad (72)$$

Substituting the proper Lyra homothetic vector (38) of the solution **(LG1)** in the MC equation (69) and making use of (72), we obtain the following constrains:

$$\begin{cases} a_2 = a_3 = a_4 = 0, \\ 2\alpha \rho(t) \exp \left[\int \frac{\theta(t)}{2} dt \right] = \left(a_1 + \alpha \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right) [\rho(t) \theta(t) - \rho'(t)], \\ 2\alpha p(t) \exp \left[\int \frac{\theta(t)}{2} dt \right] = \left(a_1 + \alpha \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right) [p(t) \theta(t) - p'(t)]. \end{cases} \quad (73)$$

Integrating the above equations with respect to t , we get:

$$\begin{cases} \rho(t) = \rho_0 e^{\int \theta(t) dt} \left(a_1 + \alpha \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right)^{-2}, \\ p(t) = p_0 e^{\int \theta(t) dt} \left(a_1 + \alpha \int \exp \left[\int \frac{\theta(t)}{2} dt \right] dt \right)^{-2}, \end{cases} \quad (74)$$

where ρ_0 and p_0 are constants of integration.

Looking at the values of $p(t)$ and $\rho(t)$ it is observed that when $\theta = \theta(t)$, the barotropic equation of state $\rho(t) = \gamma p(t)$, ($\gamma = \text{constant}$, $0 \leq \gamma \leq 1$) is satisfied for $\gamma = \rho_0/p_0$. It is interesting to see when $\theta(t) = 0$ and $\theta(t) = 2\theta_0$, the pressure and density are given respectively by:

$$\begin{cases} \theta(t) = 0 \Rightarrow \rho(t) = \rho_0 (a_1 + \alpha t)^{-2}, \quad p(t) = p_0 (a_1 + \alpha t)^{-2}, \\ \theta(t) = 2\theta_0 \Rightarrow \rho(t) = \rho_0 \theta_0^2 (\alpha + a_1 \theta_0 e^{-\theta_0 t})^{-2}, \quad p(t) = p_0 \rho(t)/\rho_0. \end{cases} \quad (75)$$

It is important to observe that when $\theta = 0$, both pressure and density of the matter field are functions of t and also depends upon the homothetic factor α . If we put the homothetic factor equal to zero then both the pressure and density will become constant. Also, for all the remaining proper Lyra homothetic vectors similar results were obtained, which we are going to omit here.

Next we put the Lyra Killing vector fields of the solution **(LG5)** in the MC equation (69), we find the following constrains:

$$\{ a_2 = a_3 = 0, \quad a_1 [\rho'(t) - \theta(t) \rho(t)] = 0, \quad a_1 [p'(t) - \theta(t) p(t)] = 0. \quad (76)$$

The above equation admits two solutions as the following:

(1): $a_1 = 0$, $\rho(t)$ and $p(t)$ are both functions of t .

(2): $a_1 \neq 0$, $\rho(t) = \rho_0 e^{\int \theta(t) dt}$ and $p(t) = p_0 e^{\int \theta(t) dt}$, where p_0 and ρ_0 are arbitrary constants.

Looking at the values of $p(t)$ and $\rho(t)$ it is observed that when $\theta = \theta(t)$, the barotropic equation of state $\rho(t) = \gamma p(t)$, ($\gamma = \text{constant}$, $0 \leq \gamma \leq 1$) is satisfied for $\gamma = \rho_0/p_0$. It is interesting to see when $\theta(t) = 0$ and $\theta(t) = 2\theta_0$, the metric functions, the pressure and density are given, respectively by:

$$\begin{cases} \theta(t) = 0 \Rightarrow U(t) = a_0, \quad W(t) = b_0 e^{-b_1 t}, \quad \rho(t) = \rho_0, \quad p(t) = p_0, \\ \theta(t) = 2\theta_0 \Rightarrow U(t) = a_0 e^{-\theta_0 t}, \quad W(t) = b_0 e^{-\theta_0 t} \exp \left[\frac{b_1}{\theta_0} e^{\theta_0 t} \right], \\ \rho(t) = \rho_0 e^{2\theta_0 t}, \quad p(t) = p_0 e^{2\theta_0 t}. \end{cases} \quad (77)$$

In all the remaining cases of Lyra Killing vector fields similar results were obtained, which we are going to omit here.

4 Conclusion

In this paper Bianchi type I spacetimes are classified according to their homothetic vectors in the context of Lyra geometry. Our classification reveals that there exist only three different possibilities where the spacetime under consideration admit proper LHVs for the special choice of the metric functions. Also the cases where Bianchi type I spacetimes do not admit proper LHVs and LHVs are just LKVs are explored. In all these cases the metric functions are obtained and it comes out that these metric functions are dependent upon the displacement vector $\theta(t)$. For the most general case **(LG1)**, we have given the Lie algebra structure, which is closed. In order to obtain homothetic vectors of Bianchi type I spacetimes in the context of general relativity, the displacement vector is set equal to zero in the HVF components of Lyra geometry and in the metric functions. In this way a classification of Bianchi type I spacetime according to homothetic vectors in general relativity is also obtained.

In some recent papers [6, 7] it was claimed that barotropic equation of state (70) never satisfies in context of Lyra geometry for plane symmetric and Bianchi type I spacetimes when the displacement vector is a function of time or when it is a constant. In order to check the consistency of those judgments, we took the matter field as a perfect fluid

and explained the matter collineation symmetry for the spacetime under consideration. We found that every LHV is also a matter collineation vector which in turn created the possibility to obtain a barotropic equation of state of the form (70). Contrary to [6, 7], in the case of Bianchi type-I spacetimes, we found that a barotropic equation of state of the form (70) is always possible to form, when the displacement vector is a function of t or is a constant.

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